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# A VARIABLE THRESHOLD MODEL FOR SIGNAL DETECTION

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## 1. Introduction

This paper deals with an analysis of some simple detection experiments in terms of Stimulus Sampling Theory (Estes (1950), Estes and Burke (1953), Estes and Suppes (1959), Atkinson and Estes (1961)). The type of study to be considered is a choice experiment for which the experimenter has established, and explained to the subject, a one-to-one correspondence between the response set $(A_1, A_2, \ldots, A_r)$ and the stimulus presentation set $(S_1, S_2, \ldots, S_r)$. On each trial a stimulus is presented and the subject attempts to identify the stimulus by making the appropriate response. For excellent reviews of theoretical and experimental research in this area see Green (1960) or Swets (1961).

We shall only consider experiments where $r = 2$; that is, on each trial either $S_1$ or $S_2$ is presented and the subject is required to make response $A_1$ or $A_2$. Also, the analysis will be restricted to procedures where the experimenter informs the subject at the end of each trial which response was correct. These two restrictions are not fundamental to the theory, but greatly simplify the presentation. Later, it will be apparent that the model can be extended to multi-stimulus problems and to procedures where information feedback is manipulated as an experimental variable.

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[1] The ideas presented in this paper have been much influenced by discussions with E. C. Carterette and R. Kinchla.

Two types of experimental procedures are to be distinguished in the analysis. We define these by example.

<u>Yes-No Procedure</u>: $S_1$ is a tone burst in a background of white noise and $S_2$ is the white noise alone. On a given trial either $S_1$ or $S_2$ is presented and the subject answers yes $(A_1)$ or no $(A_2)$ regarding the presence of the signal.

<u>Forced-Choice Procedure</u>: Two temporal intervals are defined on each trial, exactly one of which contains a signal; i.e., in one interval a tone burst in a background of white noise is presented, while in the other interval only the white noise is presented. On each trial, the subject is required to identify the interval he believes most likely to have contained the signal. Thus, $S_i (i = 1, 2)$ denotes a trial on which the signal occurred in time interval $i$ and $A_j (j = 1,2)$ denotes the subjects' selection of interval $j$ as the one containing the signal.

In this paper we shall use the identifications given in these examples. That is, for the yes-no procedure $S_1$ will always denote signal plus noise, whereas $S_2$ will denote noise alone; for the forced-choice procedure $S_1$ will denote signal plus noise in the first interval followed by noise alone in the second interval, and $S_2$ indicates noise alone in the first interval and signal plus noise in the second interval. In addition, the following notation will be used:

$S_{i,n}$ = The presentation of stimulus $S_i$ on trial $n$ of the experiment.

$A_{j,n}$ = The occurrence of response $A_j$ on trial $n$ of the experiment.

A theoretical result of particular interest in analyzing detection data deals with the relation of $Pr(A_{1,n}|S_{1,n})$ to $Pr(A_{1,n}|S_{2,n})$. For simplicity we write

$$
\begin{aligned}
p_{1,n} &= Pr(A_{1,n}|S_{1,n}) \\
p_{2,n} &= Pr(A_{1,n}|S_{2,n})
\end{aligned}
\tag{1}
$$

and when the appropriate limit exists

$$\lim_{n \to \infty} p_{i,n} = p_i \quad .$$

For the yes-no procedure $p_1$ is the asymptotic probability of a yes report when the signal is present (the likelihood of a "hit") and $p_2$ is the probability of a yes report when noise alone is presented (the likelihood of a "false alarm"). In the literature, plots of the relation of $p_2$ to $p_1$ are commonly called ROC curves, which stands for <u>receiver</u> <u>operating</u> <u>characteristic</u> curves.

In terms of our notation, two classes of variables are under the control of the experimenter: (1) the physical parameters of the stimulus presentation set, and (2) the trial-to-trial schedule for presenting stimuli. This paper deals primarily with the effects of these variables in both the yes-no and forced-choice experiments. Other factors, such as the use of special instructions designed to introduce response bias and differential monetary payoffs contingent on trial outcomes, are discussed later but are not treated in detail. The reason is that the study of such variables, within our theoretical framework, leads to models that are mathematically complex and thus warrant only limited investigation until the less complicated cases have been adequately explored.

In this paper we treat a simple probabilisitc schedule for presenting stimuli; namely

$$\Pr(S_{1,n}) = \gamma \tag{2}$$

where $\gamma$ is a constant over trials. More complex stimulus schedules can be analyzed; e.g., the stimulus presentation on trial $n$ might depend on the response on trial $n - k$, or on the stimulus on trial $n - k$, or both. However, an analysis of this simpler schedule will be sufficient to illustrate the basic concepts.

The theory generates predictions for all aspects of the subjects' response protocol (mean response probabilities, associated variances, sequential predictions such as autocorrelations, and so forth) and thereby permits a detailed treatment of individual trial-by-trial data. Most of the predictions depend on estimates of parameters that describe the stimulus situation and the hypothesized detection process. Some readers may feel that we have been too liberal in postulating parameters; however, for most applications, restrictions are appropriate that markedly reduce the number of free parameters. Further, some predictions such as the ROC curve require that only two parameters be estimated.

## 2. Axioms and Identification Rules

Readers familiar with recent developments in stimulus sampling theory will recognize that our axioms are a schematic statement of a more general theory. In this paper, we offer a simple analysis of the stimulus presentation set and postulate a learning process defined on the set of background stimuli (denoted $s_0$). In addition, two <u>perceptual states</u> (H and L) are assumed to exist and are differentiated in terms of the signal parameters associated with these states. Roughly speaking, the subject is more "alert" or "attentive" to the stimulus in state H than in state L. The particular perceptual state of the subject on any trial is a function of his history of detections and the difficulty of the task. Only two perceptual states are postulated but it will be obvious that these notions can be generalized to an n-state process.[2/]

The axioms for the model fall into two groups: The first group deals with the stimulus situation and changes in perceptual states; the second group, with the response mechanism.

## Stimulus Axioms

S1. *If the subject is in state* H *and* $S_i (i = 1,2)$ *is presented then either stimulus element* $s_i$ *will be sampled (with probability* $h_i$*), or stimulus element* $s_0$ *will be sampled.*

S2. *If the subject is in state* L *and* $S_i (i = 1,2)$ *is presented then either element* $s_i$ *will be sampled (with probability* $\ell_i$*), or element* $s_0$ *will be sampled.*

S3. $h_1 \geq \ell_1$ *and* $h_2 \geq \ell_2$ .

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2/ For an application of similar concepts to discrimination learning, see Atkinson (1960) and Atkinson (1961).

S4. _If the subject makes a response that is designated as incorrect by the experimenter, then with probability_ $\mu$ _he moves to state_ H _for the next trial; if he is already in state_ H _he remains so._

S5. _If a subject makes a response that is designated as correct by the experimenter, then with probability_ $\delta$ _he moves to state_ L _on the next trial; if he is already in state_ L _he remains so._

Response Axioms

R1. _If_ $s_i (i = 1,2)$ _is sampled on trial_ $n$ _then the_ $A_i$ _response will occur with probability_ 1 .

R2. _If_ $s_0$ _is sampled on trial_ $n$ _then the_ $A_1$ _response will occur with probability_ $\gamma_n$ _where_

$$\gamma_n = \gamma - [\gamma - \gamma_1](1 - c)^{n-1} \quad .$$

We distinguish between yes-no and forced-choice methods in terms of the signal parameters $h_i$ and $\ell_i$ . Consider first the case where the subject is in perceptual state H (i.e., $S_1$ and $S_2$ are specified by $h_1$ and $h_2$). When a signal is presented in noise we assume that the subject either detects the signal (with probability $\sigma$) or is uncertain as to whether or not the signal occurred. Similarly, when noise alone is presented we assume that the subject either detects the absence of a signal (with probability $\eta$) or is uncertain whether or not the signal occurred. The three events will be denoted as follows: s = detected signal; $\bar{s}$ = detected omission of signal; and u = uncertain. For the yes-no method the occurrence of event s is identified with the sampling of element $s_1$; $\bar{s}$ with the sampling of $s_2$; and the event u with the sampling of element $s_0$ . Hence, for the yes-no procedure

$$h_1 = \sigma \text{ and } h_2 = \eta \quad . \tag{3}$$

For the forced-choice procedure the analysis is different. Consider an $S_1$ trial--signal plus noise in the first interval followed by noise alone in the second interval. The following event sequences can occur:

(1) event $s$ occurs in the first interval and is followed by event $\bar{s}$ in the second interval--with probability $\sigma\eta$

(2) $s$ followed by $u$ --with probability $\sigma(1 - \eta)$

(3) $u$ followed by $\bar{s}$ --with probability $(1 - \sigma)\eta$

(4) $u$ followed by $u$ --with probability $(1 - \sigma)(1 - \eta)$ .

Information transmitted by either outcome 1, 2, or 3 is adequate to identify the trial, and hence the occurrence of any one of these events is associated with the sampling of element $s_1$ . If the fourth outcome occurs, we assume that element $s_0$ is sampled.[3/] Therefore, $h_1 = 1 - (1 - \sigma)(1 - \eta)$ ; by a similar argument for $S_2$ trials it can be shown that $h_2 = h_1$ . Hence, for the forced-choice method

$$h_1 = h_2 = 1 - (1 - \sigma)(1 - \eta) \quad . \tag{4}$$

Note that the signal parameter $h_1 = h_2$ for the forced-choice method is always greater than or equal to $h_1$ and $h_2$ for the yes-no procedure.

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3/ In formulating a model that also treated <u>choice time</u> it would be natural to distinguish between outcomes 1 to 3. However, for an analysis of response selection, such a distinction is not necessary. Also, note that the assignment of probabilities to the four outcomes assumes no time-order effect; i.e., no interaction between events in one temporal interval and the next. For a given experimental situation, the precision of the comparison between the forced-choice and the yes-no method will depend on the accuracy of this assumption.

By similar arguments we may express $\ell_1$ and $\ell_2$ in terms of $\sigma'$ and $\eta'$. These later parameters describe the signal when the subject is in perceptual state L .

It should be noted that the learning process postulated in Axiom R2 is highly artificial and represents only a gross approximation to current stimulus sampling models for learning. Further, for stimulus schedules other than those given by Eq. 2, it will be necessary to postulate other learning functions. For example, if we employ a contingent stimulus schedule where

$$\Pr(S_{1,n+1} | S_{1,n}) = \gamma^{(1)}$$

$$\Pr(S_{1,n+1} | S_{2,n}) = \gamma^{(2)}$$

then the function $\gamma_n = \Pr(A_{1,n} | s_{0,n})$ given in Axiom R2 would in the limit approach $\dfrac{\gamma^{(2)}}{1 - \gamma^{(1)} + \gamma^{(2)}}$. The details of how to specify a more general learning function can be obtained in Estes (1959), or Atkinson and Estes (1961). The justification for our present formulation of the learning process is that it greatly simplifies the model. We return to this point later.

## 3. Asymptotic Response Probabilities and ROC Curves

Let $C_{1,n}$ denote the event where the subject is in perceptual state H at the start of trial n ; and $C_{2,n}$ , the event of being in state L at the start of trial n . Further, we introduce the notation

$$v_n = \Pr(C_{1,n}) .$$

From axioms S1, S2, R1, and R2 it follows that

$$\Pr(A_{1,n} | S_{1,n}\ C_{1,n}) = h_1 + (1-h_1)\gamma_n$$

$$\Pr(A_{1,n} | S_{2,n}\ C_{1,n}) = (1-h_2)\gamma_n$$

$$\Pr(A_{1,n} | S_{1,n}\ C_{2,n}) = \ell_1 + (1-\ell_1)\gamma_n$$

$$\Pr(A_{1,n} | S_{2,n}\ C_{2,n}) = (1-\ell_2)\gamma_n \quad .$$

Hence, for $p_{1,n}$ and $p_{2,n}$ (as defined by Eq. 1) we obtain

$$\begin{aligned} p_{1,n} &= v_n[h_1 + (1-h_1)\gamma_n] + (1-v_n)[\ell_1 + (1-\ell_1)\gamma_n] \\ p_{2,n} &= v_n(1-h_2)\gamma_n + (1-v_n)(1-\ell_2)\gamma_n \quad . \end{aligned} \tag{5}$$

To obtain an expression for $p_{i,n}$ we need first to write $v_n$ . By axioms S4 and S5 we may prove that

$$v_{n+1} = v_n(1-\delta a_n) + (1-v_n)\mu\, b_n \tag{6}$$

where

$$a_n = \gamma[h_1 + (1-h_1)\gamma_n] + (1-\gamma)[h_2 + (1-h_2)(1-\gamma_n)]$$

$$b_n = \gamma(1-\ell_1)(1-\gamma_n) + (1-\gamma)(1-\ell_2)\gamma_n \quad .$$

A solution for this difference equation can be given but it is rather lengthy. For the moment we shall confine our attention to asymptotic predictions and therefore require only the limiting expression of $v_n$. Following the convention introduced earlier, let $v = \lim_{n \to \infty} v_n$. Then, from Eq. 6 and Axiom R2,

$$v = \frac{b}{a\varphi + b} \tag{7a}$$

where

$$\varphi = \delta/\mu \tag{7b}$$

$$a = \gamma[h_1 + (1-h_1)\gamma] + (1-\gamma)[h_2 + (1-h_2)(1-\gamma)] \tag{7c}$$

$$b = \gamma(1-\gamma)(2-\ell_1-\ell_2) \quad . \tag{7d}$$

Substituting these results in Eq. 6 and letting $\gamma_n = \gamma$, the following expressions are obtained for $p_1$ and $p_2$:

$$\begin{aligned} p_1 &= v[h_1 + (1-h_1)\gamma] + (1-v)[\ell_1 + (1-\ell_1)\gamma] \\ p_2 &= \gamma[v(1-h_2) + (1-v)(1-\ell_2)] \quad . \end{aligned} \tag{8}$$

If $\gamma = \Pr(S_{1,n})$ is permitted to vary between 0 and 1, then the ROC curve defined by the above equations is, in general, a convex function that originates at point $(0, \ell_1)$ and terminates at point $(1-\ell_2, 1)$. However, it is necessary to be more precise and distinguish three cases:

(1) If $\delta = 0$, $\mu > 0$, then asymptotically the subject is absorbed in state H and the ROC curve is given by the linear function

$$p_1 = \frac{1-h_1}{1-h_2} p_2 + h_1 \quad . \tag{9}$$

(2) If $\delta > 0$, $\mu = 0$, then asymptotically the subject is absorbed in state L and the ROC curve is

$$p_1 = \frac{1-\ell_1}{1-\ell_2} p_2 + \ell_1 \quad . \tag{10}$$

Equation 9 is represented by the upper straight line in Figure 1, and Eq. 10 by the lower line.

(3) For the general case where $\mu$, $\delta > 0$, the ROC curve is a convex function bounded between Eq. 9 and Eq. 10 that originates at point $(0, \ell_1)$ and terminates at $(1-\ell_2, 1)$. Figure 1 gives several ROC curves for the case where $h_1 = .9$, $h_2 = .5$, $\ell_1 = .2$ and $\ell_2 = .1$; the curves are distinguished by the value of $\varphi = \delta/\mu$. Successive points on each curve were generated by varying $\gamma$, the signal-presentation probability. The quantity $\varphi$ is a ratio of two non-zero probabilities and hence takes any positive number greater than zero. For $\varphi$ close to zero the ROC curve tends toward the line given by Eq. 9 in Figure 1; for large $\varphi$ the curve approaches the line given by Eq. 10.

For most experiments involving variations in $\gamma$, it seems reasonable to assume that the observed values for both $p_1$ and $p_2$ will be

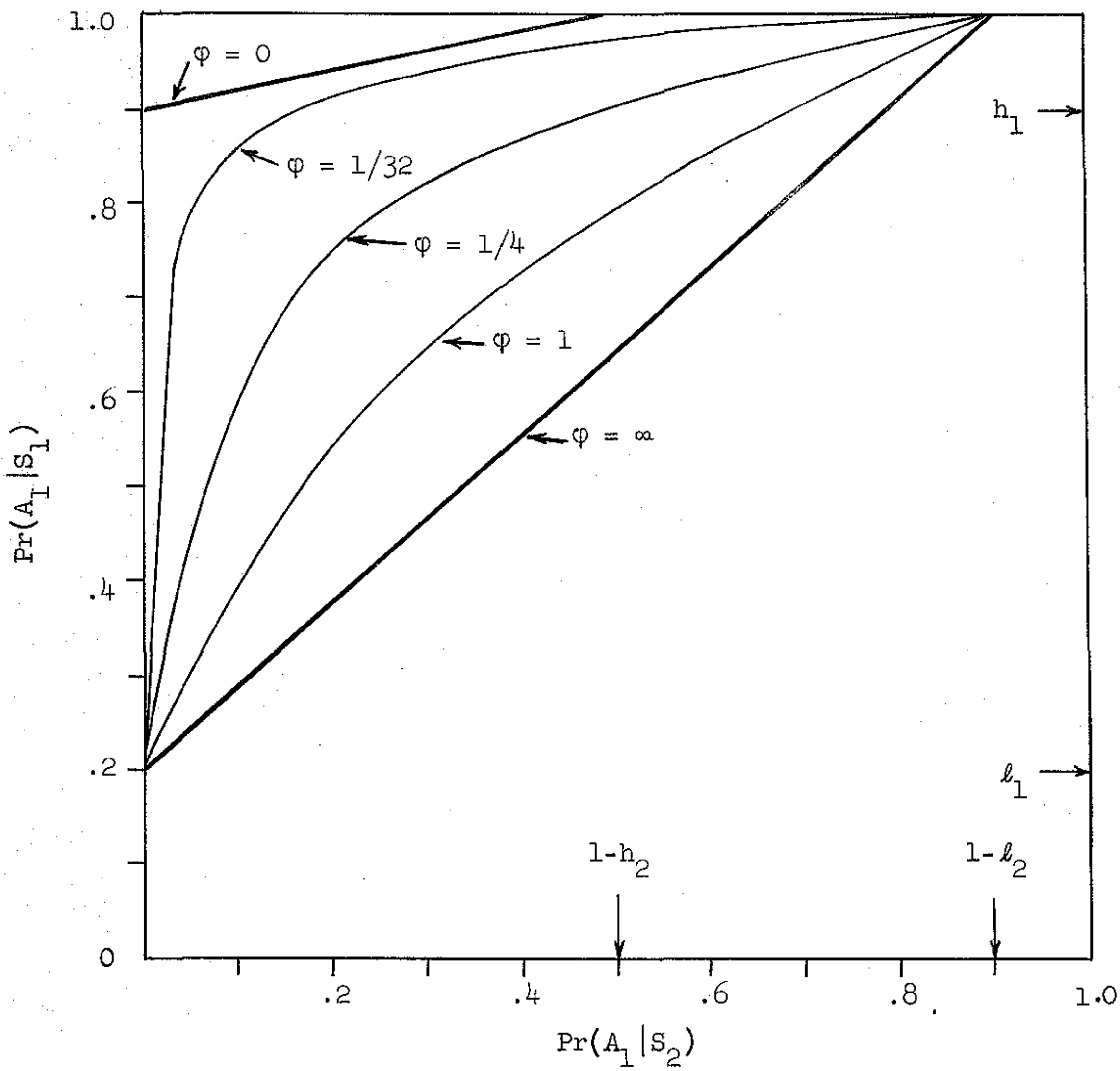
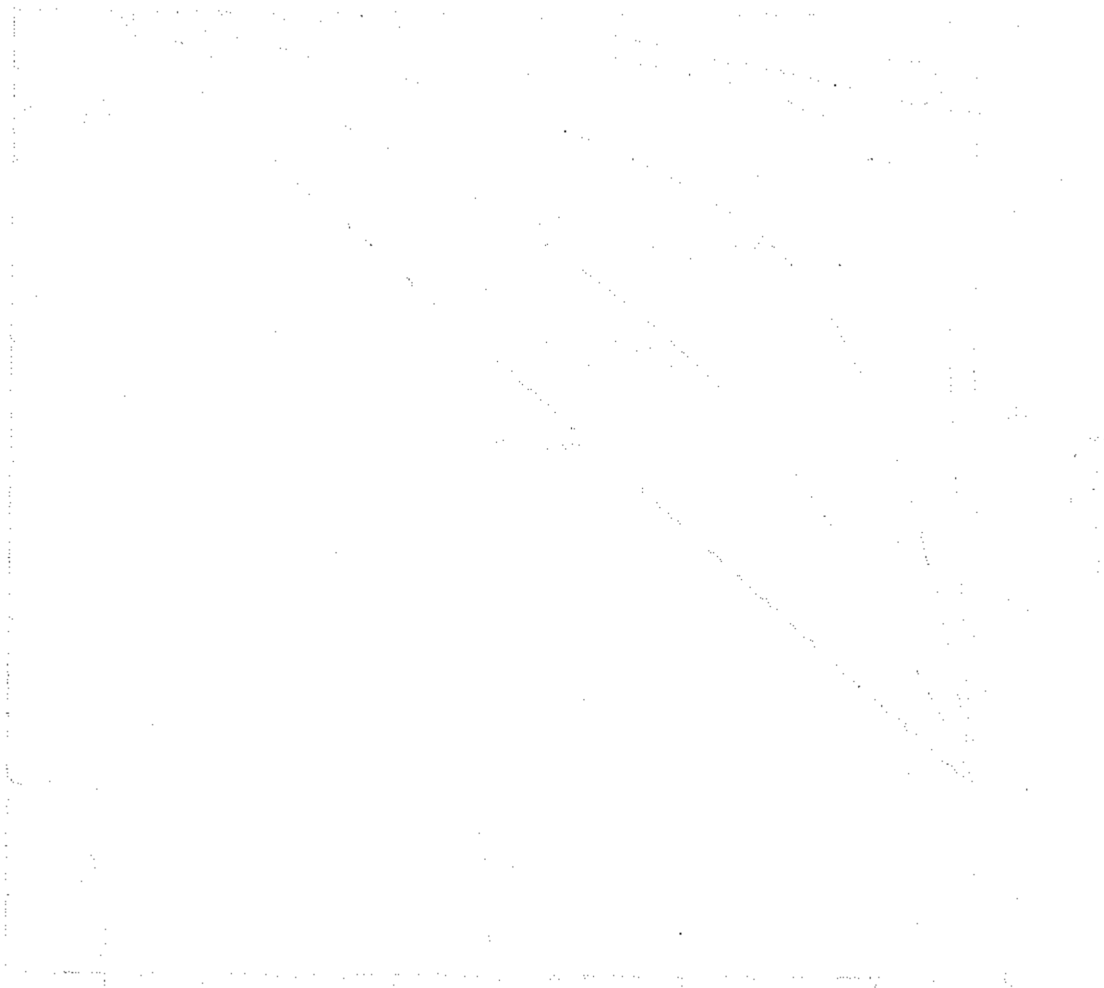



Figure 1. ROC Curves for the Case where $h_1 = .9$, $h_2 = .5$, $\ell_1 = .2$, and $\ell_2 = .1$ .



1 when $\gamma = 1$ and 0 when $\gamma = 0$. In theory, the prediction

$$p_1 = p_2 = \begin{cases} 1, & \text{if } \gamma = 1 \\ 0, & \text{if } \gamma = 0 \end{cases}$$

requires that $\ell_1 = \ell_2 = 0$. Given this restriction the ROC curve traces out a convex function running from 0 to 1 on both coordinates.

Most of the experimental work on signal detection suggests that the ROC curve originates at 0 and terminate at 1. Consequently, in the remainder of this paper we require $\ell_1 = \ell_2 = 0$. Given this assumption Eq. 7 and 8 may be rewritten as follows:

$$\frac{1}{v} = 1 + \varphi\left\{\frac{\gamma[h_1 + (1-h_1)\gamma] + (1-\gamma)[h_2 + (1-h_2)(1-\gamma)]}{2\gamma(1-\gamma)}\right\}$$

$$p_1 = vh_1 + \gamma(1-vh_1) \tag{11}$$

$$p_2 = \gamma(1-vh_2) \quad .$$

We now compare ROC curves for the forced-choice method and the yes-no method. By an earlier argument we established that, for the yes-no method

$$h_1 = \sigma$$

$$h_2 = \eta \quad .$$

While for the forced-choice method

$$h_1 = h_2 = \sigma + \eta(1-\sigma) \quad .$$

Thus, to fit an ROC curve for the forced-choice procedure only two parameters are needed ($h$ and $\varphi$); for the yes-no experiment three parameters are required ($h_1$ , $h_2$ and $\varphi$) . If the same physical stimuli are used in a yes-no experiment and in a forced-choice experiment (i.e., $\sigma$ and $\eta$ are the same for both experiments) and we assume that variables related to $\varphi$ are held constant for both procedures, then the theory predicts that the ROC curve generated by the forced-choice group will be above the ROC curve for the yes-no group (except at (0,0) and (1,1) where they are equal). Also, the ROC curve for the forced-choice method is symmetric about the main diagonal from point (0,1) to (1,0); for the yes-no method the ROC curve may be symmetric about the main diagonal (if $\sigma = \eta$) ; skewed to the left (if $\sigma > \eta$) ; or skewed to the right (if $\sigma < \eta$) .

To illustrate these remarks we compute some ROC curves for the forced-choice and the yes-no method. Let $\sigma = .75$ , and $\eta = .50$ . Then, for the forced-choice condition $h_1 = h_2 = .875$ , whereas for the yes-no condition $h_1 = .75$ , $h_2 = .50$ . Figure 2 gives the ROC curves for the forced-choice and yes-no methods for several different values of $\varphi$ . As noted before, when $\varphi \to 0$ the ROC curve approaches the line $p_1 = (.50)p_2 + .75$ for the forced-choice method and the line $p_1 = p_2 + .875$ for the yes-no method. As $\varphi \to \infty$ , the ROC curves for both methods approach the line $p_1 = p_2$ .

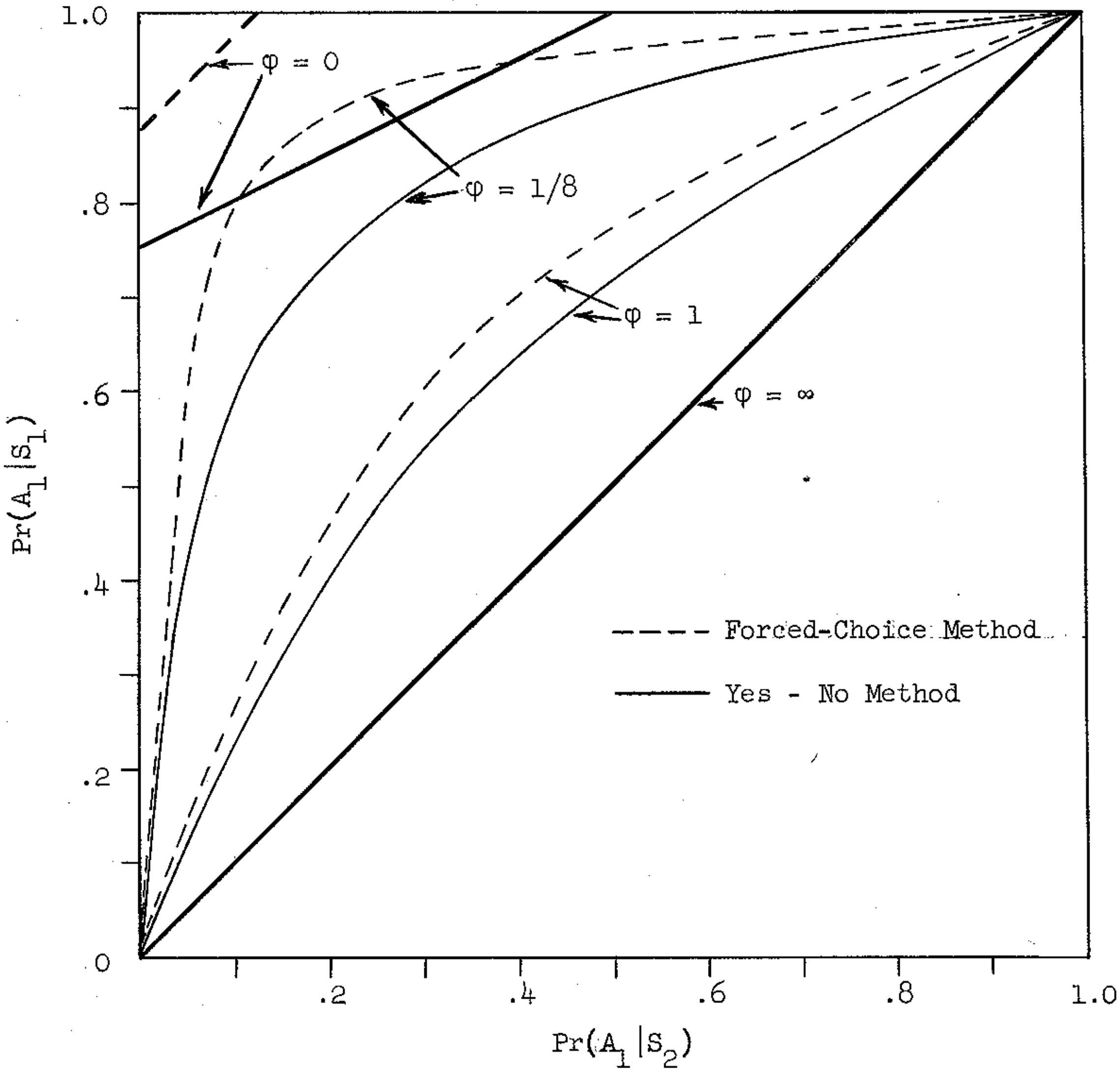
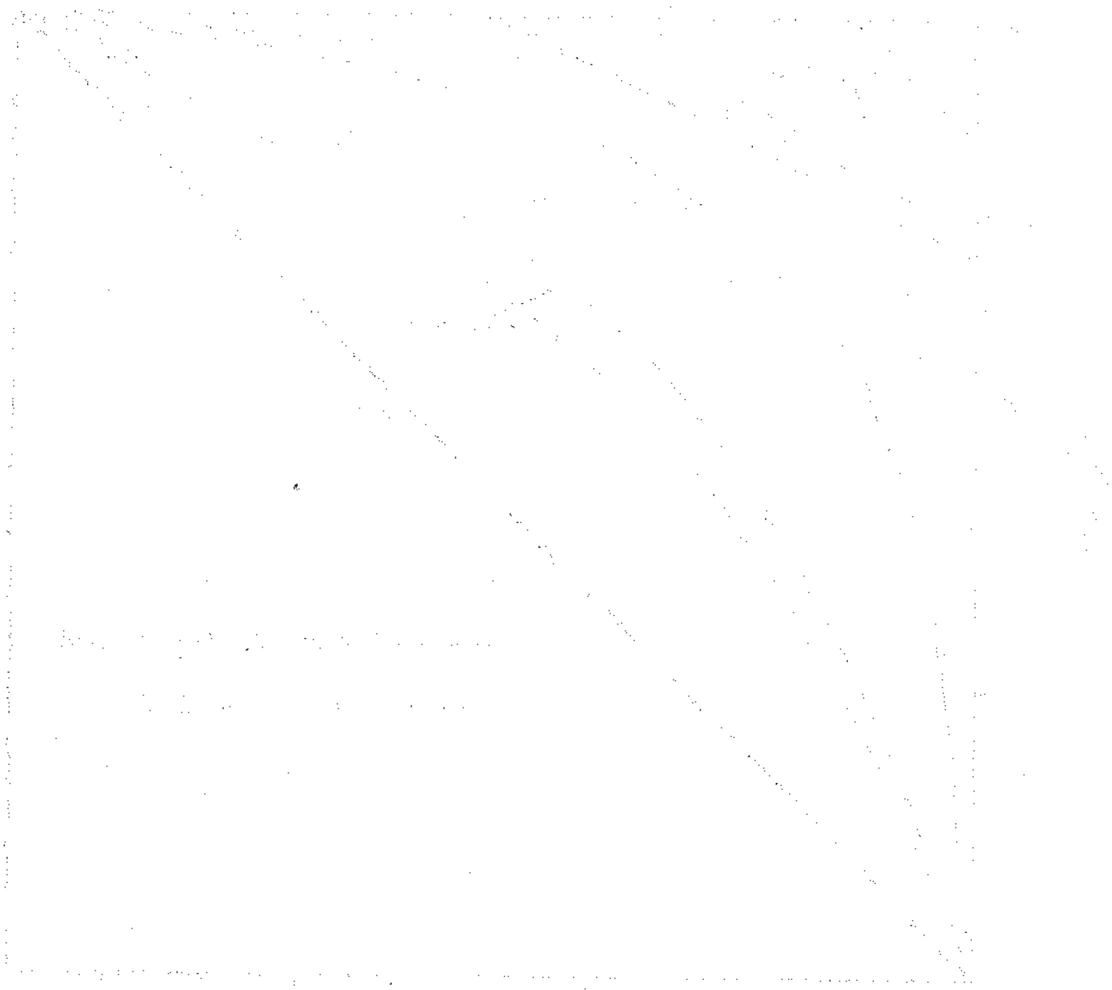



Figure 2. ROC Curves for the Forced-Choice Method and the Yes-No Method.



## 4. Sequential Predictions

It has long been recognized that rather complex trial-to-trial dependencies are involved in most psychophysical data. Recently, some very striking sequential effects have been reported by Carterette (1962) in a signal detection experiment. In this section we derive some sequential predictions, having selected those quantities that are particularly useful in making estimates of $\mu$ and $\delta$. The reader is referred to Suppes and Atkinson (1960; Chapter 2) for a discussion of appropriate estimation procedures.

We shall examine predictions regarding the influence of stimulus and response events on trial $n$ as they affect the response on trial $n + 1$. Specifically,

$$\Pr(A_{1,n+1} | S_{1,n+1} A_{i,n} S_{j,n}) \qquad i, j = 1, 2 .$$

That is, the probability of an $A_1$ response to $S_1$ conditionalized on the various outcomes of the preceding trial. Consider first $\Pr(A_{1,n+1} | S_{1,n+1} A_{1,n} S_{1,n})$ which, by elementary probability considerations, can be written as follows:

$$\Pr(A_{1,n+1} | S_{1,n+1} A_{1,n} S_{1,n}) = \frac{\Pr(A_{1,n+1} S_{1,n+1} A_{1,n} S_{1,n})}{\Pr(S_{1,n+1} A_{1,n} S_{1,n})} . \tag{12}$$

Now, we need expressions for the numerator and denominator on the right-hand side of the above equation. First, note that the denominator may be expanded:

$$\Pr(S_{1,n+1}\ A_{1,n}\ S_{1,n}) = \Pr(S_{1,n+1}|A_{1,n}\ S_{1,n})\ \Pr(A_{1,n}|S_{1,n})\ \Pr(S_{1,n}) .$$

But $\Pr(S_{1,n+1}|A_{1,n}\ S_{1,n}) = \Pr(S_{1,n}) = \gamma$ and by Eq. 1 $\Pr(A_{1,n}|S_{1,n}) = p_{1,n}$ . Hence

$$\Pr(S_{1,n+1}\ A_{1,n}\ S_{1,n}) = \gamma^2\ p_{1,n} . \tag{13}$$

Similarly, for the numerator of Eq. 12 we write

$$\sum_{i,j} \Pr(A_{1,n+1}\ S_{1,n+1}\ C_{i,n+1}\ A_{1,n}\ S_{1,n}\ C_{j,n}) \tag{14}$$

$$= \sum_{i,j} \Pr(A_{1,n+1}|S_{1,n+1}\ C_{i,n+1})\gamma\ \Pr(C_{i,n+1}|A_{1,n}\ S_{1,n}\ C_{j,n})$$

$$\cdot \Pr(A_{1,n}|S_{1,n}\ C_{j,n})\gamma\ \Pr(C_{j,n}) .$$

By definition $\Pr(C_{1,n}) = v_n$ , and by Eq. 5

$$\Pr(A_{1,n+1}|S_{1,n+1}\ C_{i,n+1}) = \begin{cases} h_1 + (1-h_1)\gamma_{n+1} , & \text{for } i = 1 \\ \gamma_{n+1} , & \text{for } i = 2 \end{cases}$$

Further, by Axioms S4 and S5

$$\Pr(C_{i,n+1} | A_{1,n}\ S_{1,n}\ C_{j,n}) = \begin{cases} 1-\delta\ , & \text{for } i=1\ ,\ j=1 \\ \delta\ , & i=2\ ,\ j=1 \\ 0\ , & i=1\ ,\ j=2 \\ 1\ , & i=2\ ,\ j=2 \end{cases}$$

Hence, carrying out the summation in Eq. 14 we obtain

$$\gamma^2 \left\{ v_n [h_1 + (1-h_1)\gamma_n][\delta\gamma_{n+1} + (1-\delta)(h_1 + \gamma_{n+1} - h_1\gamma_{n+1})] + (1-v_n)\gamma_n\gamma_{n+1} \right\} . \quad (15)$$

Dividing Eq. 15 by Eq. 13 yields the desired expression for $\Pr(A_{1,n+1} | S_{1,n+1}\ A_{1,n}\ S_{1,n})$ . For most applications we deal with asymptotic data; that is, for trial sequences where $n$ is large. Under these conditions $\gamma_n \to \gamma$ , $v_n \to v$ , and $p_{1,n} \to p_1$ ; as a result, much simplification is possible. We now rewrite Eq. 12 for the case where $n \to \infty$ , and also present expressions for the other asymptotic sequential effects. Following our earlier convention, the subscripts $n$ and $n+1$ will be deleted to indicate limiting quantities but are implicit in the ordering. Further, to simplify the expressions we define $\pi = h_1 + (1-h_1)\gamma$ . Then

$$
\begin{aligned}
\Pr(A_1 | S_1 A_1 S_1) &= \frac{1}{p_1} \left\{ v\pi[\delta\gamma + (1-\delta)\pi] + (1-v)\gamma^2 \right\} \\
\Pr(A_1 | S_1 A_2 S_1) &= \frac{1-\gamma}{1-p_1} \left\{ v(1-h_1)\pi + (1-v)[\mu\pi + (1-\mu)\gamma] \right\} \\
\Pr(A_1 | S_1 A_1 S_2) &= \frac{\gamma}{p_2} \left\{ v(1-h_2)\pi + (1-v)[\mu\pi + (1-\mu)\gamma] \right\} \\
\Pr(A_1 | S_1 A_2 S_2) &= \frac{1}{1-p_2} \left\{ v[h_2 + (1-h_2)(1-\gamma)][\delta\gamma + (1-\delta)\pi] + (1-v)(1-\gamma)\gamma \right\} .
\end{aligned} \tag{16}
$$

Any other sequential prediction can be derived but the above are of particular interest with regard to estimation methods and illustrate the type of prediction that is possible.

5. Discussion

For our model, the ROC curve is specified by the parameters $\varphi$, $h_1$ and $h_2$, with $h_1$ being equal to $h_2$ in the forced-choice procedure. In theory, $h_1$ and $h_2$ are measures of $S_1$ and $S_2$ and depend only on the physical parameters describing the stimulus presentation set. It is assumed that other variables such as stimulus presentation schedules, variations in instructions, monetary payoffs, and experimental design have no affect on the value of $h_i$. Consequently, given a specific stimulus set, differences in the ROC curves from one experimental routine to another are to be represented in terms of variations in $\varphi$. Roughly speaking, one can argue that experimental manipulations that increase a subject's motivation or interest in the detection task will give rise to both an increase in $\mu$ and a decrease in $\delta$; i.e., tend to decrease the value of $\varphi$. It was indicated earlier that as $\varphi$ decreases the ROC curve tends to approach the function

$p_1 = \frac{1-h_1}{1-h_2} p_2 + h_1$ ; whereas, if $\varphi$ increases the ROC curve approaches the function $p_1 = p_2$ . In addition, predicted differences between the ROC curve for the forced-choice and yes-no method increase as $\varphi$ becomes small. Consequently, by manipulating experimental variables related to $\varphi$ one should be able to modify the convexity of the ROC curve, and also vary the amount of difference between ROC curves obtained under forced-choice and yes-no conditions.

The use of monetary payoffs may be one technique for manipulating $\varphi$ but the procedure suggests certain complications. Recall that we have postulated a learning function that in the limit matches the likelihood of presenting an $S_1$ stimulus; i.e., $Pr(A_{1,n}|s_{0,n}) \rightarrow \gamma$. For verbal learning experiments that do not involve monetary payoffs (Estes and Straughan (1954); Detambel (1955); Grant, Hake and Hornseth (1951); and others) an asymptotic matching assumption gives a fairly adequate description of the data; however, the use of monetary payoffs may cause the subject to deviate from matching behavior in the direction of a more optimal strategy. If the introduction of monetary rewards in a signal detection experiment has a similar effect on the hypothesized learning process associated with element $s_0$ , then it may be necessary to postulate a learning function other than the one given in Axiom R2. There are a number of theoretical developments in the literature that are relevant to this problem (e.g., Estes (1962), Atkinson (1962), Siegel (1961)) and any of these proposals could be used in place of the functions given in Axiom R2. For example, following Atkinson's formulation one might assume that $Pr(A_{1,n}|s_{0,n})$ in the limit approaches

$$\frac{\gamma^2[\gamma + (1-\gamma)\xi]}{\gamma^3 + (1-\gamma)^3 + \gamma(1-\gamma)\xi}$$

where $\xi$ is a utility measure associated with the payoff function. Such modifications may turn out to be necessary, but it also may be that the effects of monetary payoff can be accounted for in terms of $\varphi$ alone. An answer to this question will depend on a detailed inspection of sequential data and cannot be obtained by an analysis of gross statistics like $\hat{p}_1$ and $\hat{p}_2$.

The sequential effects predicted by this model are principally due to trial-to-trial changes in perceptual states. Another source of variability in signal detection experiments may result from trial-to-trial fluctuations in the learning process associated with background stimuli. In our model a learning process is assumed but we do not allow for trial-to-trial learning effects; this fact becomes clear when one observes that in the limit $Pr(A_{1,n}|s_{0,n})$ is a fixed number $\gamma$ and not a distribution with expectation $\gamma$. It is the absence of these sequential effects in the learning process that elicited our earlier comment on the artificial nature of this aspect of the model. If it turns out that learning effects, other than those incorporated in Axiom R2, are important in accounting for sequential phenomena then it will be necessary to postulate a more general learning process. We have formulated such a model: it involves two additional axioms dealing with the conditioning of the $s_0$ element and a restatement of Axiom R2. They are as follows:

C1. _On every trial element_ $s_0$ _is conditioned to either_ $A_1$ _or_ $A_2$ .

C2. _If_ $s_0$ _is sampled on a trial, it becomes conditioned with probability_ $c$ _to the response that was correct on the trial_.

R2.* _If_ $s_0$ _is sampled, then the response to which_ $s_0$ _is conditioned will occur_.

The mathematical problems introduced by these additional assumptions makes an analysis of the model more difficult. The response probabilities are functions defined on a 4-state Markov chain, where the states of the chain are unobservable. We have investigated ROC curves for a number of cases and they conform very closely to the same functions derived from the model presented in this paper. In fact, it seems reasonable to suppose that for grosser predictions, such as $p_1$ and $p_2$ , the agreement between the two models will be very close. Thus, if it becomes necessary to modify the axioms along these lines, then the equations given in this paper may be viewed as a simple device for computing the grosser predictions of the general theory.

There are a number of special topics that have not been discussed. Of interest, is the relation of our model to theories of discrimination learning (particularly, Burke and Estes (1957), Restle (1955) and Atkinson (1960)); the effect of blank trials in a forced-choice procedure; the effect of incorrect information; extension of the model to account for choice-time measures; and the extension of the model to multi-stimulus-response problems. These problems can be formulated and analyzed within the framework of our model, and will be treated in later papers.

In summary, it seems reasonable to describe the model as an example of a variable-threshold theory of detection. We have postulated not one, but two thresholds. These thresholds are defined via the construct of a perceptual state. From trial-to-trial changes occur in the perceptual state of a subject, and the changes depend in a rather intricate way on the difficulty of the psychophysical task and the subjects' short-term history of detections. The perceptual states are not observable, but they are functionally related to response probabilities and consequently permit the experimenter to make a detailed analysis of all aspects of a subject's response protocol.

## APPENDIX

For those interested, some mathematical results will be presented on the detection process proposed in the last section; i.e., results for the model defined by axioms S1-S5, C1, C2, R1 and R2*. For simplicity, we consider the case where $\ell_1 = \ell_2 = 0$ and $\Pr(S_{1,n}) = \gamma$. At the start of any trial, the subject is in one of the following four states: $1 = \langle H, 1 \rangle$, $2 = \langle H, 2 \rangle$, $3 = \langle L, 1 \rangle$, $4 = \langle L, 2 \rangle$. The first member of the ordered pair indicates the perceptual state (H or L) and the second component, the conditioning of the $s_0$ element ($A_1$ or $A_2$). From the axioms, it can be shown that the sequence of random variables that take these four states as values over trials of an experiment is a Markov chain. This means, among other things, that a transition matrix $P = [p_{ij}]$ may be defined, where $p_{ij}$ is the probability of being in state $j$ on trial $n + 1$ given that the subject was in state $i$ on trial $n$. The detection process is completely characterized by the transition probabilities and the initial probability distribution on the four states. The $p_{ij}$'s can be easily derived (see Atkinson (1960) for an illustration of the methods involved) and are as follows:

$$p_{11} = \gamma(1-\delta) + (1-\gamma)[h_2(1-\delta) + (1-h_2)(1-c)]$$

$$p_{12} = (1-\gamma)(1-h_2)c$$

$$p_{13} = \gamma\delta + (1-\gamma)h_2\delta$$

$$p_{14} = 0$$

$$p_{21} = \gamma(1-h_1)c$$

$$p_{22} = \gamma[h_1(1-\delta) + (1-h_1)(1-c)] + (1-\gamma)(1-\delta)$$

$$p_{23} = 0$$

$$p_{24} = \gamma h_1\delta + (1-\gamma)\delta$$

$$p_{31} = (1-\gamma)\mu(1-c)$$

$$p_{32} = (1-\gamma)\mu c$$

$$p_{33} = \gamma + (1-\gamma)(1-\mu)(1-c)$$

$$p_{34} = (1-\gamma)(1-\mu)c$$

$$p_{41} = \gamma\mu c$$

$$p_{42} = \gamma\mu(1-c)$$

$$p_{43} = \gamma(1-\mu)c$$

$$p_{44} = \gamma(1-\mu)(1-c) + (1-\gamma) \quad .$$

Let $u_{i,n}$ be the probability of being in state $i$ at the start of trial $n$ and when, the appropriate limit exists, $\lim_{n \to \infty} u_{i,n} = u_i$ . Then for the row matrix $U_n = [u_{1,n}, u_{2,n}, u_{3,n}, u_{4,n}]$ we have that

$$U_{n+1} = U_n P$$

and, in general,

$$U_n = U_1 P^{n-1} .$$

(For a discussion of methods to obtain an explicit expression for $u_{i,n}$ see Suppes and Atkinson (1960)).

Experimentally, it is not possible to identify individual states of the process on a given trial. That is, knowing which stimulus ($S_1$ or $S_2$) and response ($A_1$ or $A_2$) occurred does not provide enough information to identify the state. For example, if $S_1$ is presented and $A_1$ occurs it is possible for the subject to have been in any one of the following states: $< H, 1 >$, $< H, 2 >$ or $< L, 1 >$. However, observable response probabilities are well-defined in terms of these unobservable states. By axioms R1 and R2* we have (for $\ell_1 = \ell_2 = 0$)

$$p_{1,n} = u_{1,n} + h_1 u_{2,n} + u_{3,n}$$

$$p_{2,n} = (1-h_2)u_{1,n} + u_{3,n} \quad .$$

As indicated earlier, the ROC curve specified by these equations has the same general properties as our simpler model. Specificially, (i) if $\delta = 0$, $\mu > 0$, then the ROC curve is defined by the linear equation $p_1 = \frac{1-h_1}{1-h_2} p_2 + h_1$; (ii) if $\delta > 0$, $\mu = 0$, then the curve is simply $p_1 = p_2$ and (iii) for $\delta > 0$, $\mu > 0$, the ROC curve is a convex function running from 0 to 1 on both coordinates and bounded between the functions $p_1 = p_2$ and $p_1 = \frac{1-h_1}{1-h_2} p_2 + h_1$.

To illustrate another feature of the model, some asymptotic sequential predictions are displayed that may be compared with Eq. 16. Namely,

$$\Pr(A_1|S_1A_1S_1) = \frac{1}{p_1}\left[u_1 + u_3 + u_2(1-\delta)h_1^{\ 2}\right]$$

$$\Pr(A_1|S_1A_2S_1) = \frac{1}{1-p_1}\left\{u_2(1-h_1)[c + (1-c)h_1] + u_4[c + \mu(1-c)h_1]\right\}$$

$$\Pr(A_1|S_1A_1S_2) = \frac{1}{p_2}\left\{u_1(1-h_2)[ch_1 + (1-c)] + u_3[c\mu h_1 + (1-c)]\right\}$$

$$\Pr(A_1|S_1A_2S_2) = \frac{1}{1-p_2}\,[u_1h_2 + u_2(1-\delta)h_1] \quad .$$

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